Finite Apertures, Interfaces &
Point Spread Functions (PSF) &
Convolution

Tutorial by Jorge Márquez Flores
Figure 1. Visual acuity (focus) depends on aperture (iris), lenses, focal point location, the geometry of the light-ray paths and the shape of the eye.
Pinhole Camera Model (Idealized, Mathematical Concept)

Geometrical (or ray)-optics, image formation with a pinhole camera with an infinitesimal aperture, infinitesimal sampling time, infinite bandwidth and non-physical light rays
**Pinhole Camera Model** (extrapolable to other forms of energy)

Finite window (size of aperture = 1/f) + finite exposure

Screen (as in the eye retina)  With smaller 1/f:

\[ I(x,y) \]

**Collimation**

\[ I(x,y) \ast \text{PSF} + \text{noise} \]

Geometrical-optics projection (ondulatory-optics ⇒ diffraction ⇒ adds blur!)

**Compromise:** small 1/f ⇒ less blur  BUT  low intensity (few photons)

large 1/f ⇒ more blur  BUT  high intensity (many photons)

† In most imaging systems the *collimation width* should be ideally 0; but in other cases it should be the largest possible for minimizing angular dispersion (e.g., *gamma cameras*)

‡‡ The coordinate inversion \((x, y) \rightarrow (-x, -y)\) induced by the pinhole is already considered by the convolution (*).
Shrinking the Aperture: down to the Diffraction Limit
The optimal pinhole diameter was first attempted by Jozef Petzval. The crispest image is obtained using a pinhole size determined by the formula: \( d = \sqrt{2f\lambda} \), where \( d \) is pinhole diameter, \( f \) is focal length (distance from pinhole to image plane) and \( \lambda \) is the wavelength of light. For standard black-and-white film, a wavelength of light corresponding to yellow-green (550 nm) should yield optimum results. For a pinhole-to-film distance of 1 inch (25 mm), this works out to a pinhole 0.17 mm in diameter. For 5 cm, the appropriate diameter is 0.23 mm.
Pinhole versus Lens

1) Image formed on the film is not tack sharp.
2) Only a small amt of light enters through the pinhole.
3) Diffraction effects.

Pinhole Camera

1) A lens focuses the light rays to produce a sharp image on the film or sensor.
2) A lens admits larger amounts of light permitting proper exposures in a small fraction of a second.

Camera with lens to focus image.
Finite Aperture and the Interface Effect (General)

Finite\textsuperscript{+} window (size) for sample
Finite time for sampling + delays
Finite spectral band (window)
Limited number of channels and
finite channel capacity
Finite sensor resolution & precision
(discretization and quantization)
Finite impedance (\(\Rightarrow\) losses)
Finite isolation (perturbable)

Finite Camera Model

Pinhole Model

Noise and metamerism

Emission or diffuse refraction

Attenuation, filtering, distortion

output=\text{input}\times\text{PSF} + \text{noise} + \text{delays}

Impedance: opposition to flow

\text{Impedance: non infinite or not null (=not zero)}

Effects on the measurant from sensor & observer

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A photon’s life choices \( (\text{as } f(\nu, (\theta, \varphi))) \)

- Emission
- Absorption
- Reflection
- Transmission (Transparency/Opacity)
- Refraction
- Diffusion
- Fluorescence
- Phosphorescence
- Subsurface scattering
- Interreflection
- Diffraction
- Interference (constructive/destructive superposition)
- Polarization, ...
“c” (at right) denotes the circle of confusión (CoC) for a out-of-focus subject at distance $S_2$ when the lens focus is at $S_2$. The CoC diameter is calculated from the lens and ray diagram; The dashed line marks an auxiliary blur circle “C” (at left) in the object plane and helps to obtain the calculation.
Finite Aperture/Lens PSF (in imaging systems)

**PSF:** Point Spread Function (or Impulse Response)

**Input:** Impulse function $\delta(x,y)$

**Output:** $PSF(x,y) = h(x,y) = T(\delta(x,y))$.

FWHM: Full Width at Half Maximum

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Output: $PSF(x,y) = h(x,y) = T(\delta(x,y))$.

**FWHM:** Full Width at Half Maximum

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**LSF: Line Spread Function (or Slit or Line Response)**

**Input:** Slit function $\delta(x)$  

**Output:** $LSF(x)$
**ESF:** Edge Spread Function (or *Step Response*)

\[ LSF(x) = \frac{\partial}{\partial x} ESF(x, y) \]

**Warning note:** Differentiating a signal enhances noise while integrating a signal attenuates noise.
**Linear Shift-Invariant Model of Degradation**

*Effects altering a measurement or acquisition*

\[ \mathcal{X}_{obs} = \mathcal{X}_{in} \ast \psi + \eta \]

**\( \mathcal{X}_{in} \)** **Entry** (data, signal/sequence, vector, image, volume, video, etc.).

**\( \mathcal{X}_{obs} \)** **Observation** (sample/measurement, output/response of the acquisition system).

**\( \psi \)** **PSF** of the sensor (*Point Spread Function \approx \text{aperture function})*.

**\( \eta \)** **Additive noise** in the \( \mathcal{X}_{obs} \) domain (sensor). May be: \[ \eta = \eta_{in} \ast \psi' + \eta' \]

**\( \ast \)** **Convolution operator**, in the \( \mathcal{X}_{in} \) domain (spatial and/or temporal).

\[ \varepsilon = \left( \mathcal{X}_{obs} - \mathcal{X}_{in} \right) \text{ Diffferential Error} \] (simplification: same domains).

\[ \mathcal{X}_{in} \quad \mathcal{X}_{in} \ast \psi + \eta_{out} \quad (\mathcal{X}_{in} + \eta_{in}) \ast \psi \]
A complex system may be characterized by several PSFs in cascade. An example is the Positron Emission Tomography imaging. There is a PSF\textsubscript{process} of the physical imaging process (image formation from isotope absorption) and a PSF\textsubscript{acquisition} of the imaging system (image production). The source itself may contribute with a third intrinsic, blurring PSF. The convolution model is only valid for Linear Shift Invariant systems. In the spatial-frequency domain, each cascade $MTF_i \propto \mathcal{F}(PSF_i)$, $i=1,2,...$, is multiplied.

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<table>
<thead>
<tr>
<th>Gaussian Kernels</th>
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<tr>
<td><strong>Isotropic Gaussian Kernel 1D</strong> centered at ( x_0 )</td>
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<tr>
<td><strong>Isotropic Gaussian Kernel 2D</strong> (or bivariate) centered at ((0,0))</td>
</tr>
<tr>
<td><strong>Isotropic Gaussian Kernel 3D</strong> (or trivariate) centered at ((0,0,0))</td>
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**Anisotropic Gaussian Kernel, diagonal covariance** \( \Sigma = \|\tilde{\sigma}\|^2 \), center at \( \tilde{x}_0 = (x_0, y_0, z_0) \)

\[
G_{\sigma, \tilde{x}_0}(\tilde{x}) \triangleq \frac{1}{\left(2\pi\right)^{3/2}} \frac{1}{\sigma_x \sigma_y \sigma_z} \exp\left[-\left(\frac{(x-x_0)^2}{2\sigma_x^2} + \frac{(y-y_0)^2}{2\sigma_y^2} + \frac{(z-z_0)^2}{2\sigma_z^2}\right)\right]
\]

**Note**: \( \tilde{\sigma} = (\sigma_x^2, \sigma_y^2, \sigma_z^2) \)
Figure 7. The **bell-shaped Gaussian function (or normal) in 1D**. Dark blue is less than one standard deviation \( \sigma \) (“a sigma”) from the mean \( \mu \) (of a population, if the curve describes a probabilistic distribution). Note that here the mean \( \mu \) is the offset \( x_0 \) used beforehand. For the normal distribution, \( \pm \)one sigma \( \sigma \) accounts for about 68% of the population, while \( \pm \)two sigmas from the mean (medium and dark blue) account for about 95%, and \( \pm \)three sigmas (light, medium, and dark blue) account for about 99.7%.
Generalized nD Anisotropic Gaussian Kernel, non-diagonal, covariance $\Sigma$, center at $\mathbf{x}_0$

$$G_{\Sigma,\mathbf{x}_0}(\mathbf{x}) \triangleq \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp \left\{ -\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T \Sigma^{-1} (\mathbf{x} - \mathbf{x}_0) \right\}$$

Notation: vector $\mathbf{x}_0 = (x_0, y_0, z_0)$ has been changed to column vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\bar{\sigma} = (\sigma_x^2, \sigma_y^2, \sigma_z^2)$

Figure 8. (Left) Anisotropic Gaussian function as 2D image. (Right) Intensity surface plot.
Figure 9. (Left) Different mathematical models of 2D Point Spread Functions (PSF) in the spatial domain and (right) their respective Fourier Transforms (frequency domain).
Impulse Response Superposition in Linear Shift-Invariant Systems (1D)

Figure 10. (a) Intensity samples of a signal \( f(x) \) (or a scan line from an image feature), decomposed into (Dirac Delta) \( \delta \) impulse functions which are then transformed by the system or processor (b) into Point Spread Functions (PSF, the impulse response) and added together (=linear superposed) to give the shaded profile \( g(x) \), (signal or an image profile). Such superposition is possible if the PSF does not change along the data domain (=shift invariance).

A decomposition such as (a) is expressed in general as: 
\[
f(x) = \int_{-\infty}^{\infty} f(x-\xi)\delta(\xi)d\xi = \int_{-\infty}^{\infty} \delta(x-\xi)f(\xi)d\xi \]
Impulse Response Superposition in *Linear Shift-Invariant Systems*

\[ h(\xi) = T(\delta(\xi)) \]: impulse response (PSF) of a system or processor \( T \)

![Impulse Response Graph](image)

**Figure 11.** The right line over \( x \) is: the contribution of the *impulse response* \( h(\xi) \) or PSF, centered at the point \( \xi \), to the signal point (or image pixel, in one scan line) at position \( x \). Details smaller than the *Full Width at Half Maximum* (FWHM) of the PSF will be blurred and not resolved, as can be seen in Figure 10-b.

*Shift invariance* is defined by the condition: \( h(\xi, x) = h(\xi - x) \), "same local \( h \)"
Impulse Response Superposition in *Linear Shift-Invariant Systems*

Impulse properties for any input function \( f \) with \( \delta(x) = \) identity element of convolution:

\[
\begin{align*}
  f(x) &= \int_{-\infty}^{\infty} f(x - \xi) \delta(\xi) d\xi = \int_{-\infty}^{\infty} \delta(x - \xi) f(\xi) d\xi \\
  &= (f * \delta)(x) = (\delta * f)(x)
\end{align*}
\]

\[
\begin{align*}
  g(x) &= T(f)(x) \\
  &= T \int_{-\infty}^{\infty} f(\xi) \delta(x - \xi) d\xi \\
  &= \int_{-\infty}^{\infty} T\left( f(\xi) \delta(x - \xi) \right) d\xi \\
  &= \int_{-\infty}^{\infty} f(\xi) T(\delta(x - \xi)) d\xi \\
  &= \int_{-\infty}^{\infty} f(\xi) h(x - \xi) d\xi \\
  &= (f * h)(x)
\end{align*}
\]

LSI system

**Notation**

\[
\int_{-\infty}^{\infty} f(\xi) d\xi = \int_{-\infty}^{\infty} \delta(\xi) f(\xi) d\xi = (f * \delta)(x) = (\delta * f)(x)
\]

**Rewrite the above formulation for 2D and its discrete version** \( f_{ij} \)
The Continuous (Image) Processor

\( T(\bullet) \) is in general the transfer function of a system, a sensor, an instrument, actuator or for a model of any of the latter. It can also characterize a processing method, a filter or algorithm.

A two-dimensional **continuous image processor** is considered to be a transformation \( T(\bullet) \) of an **input** 2D continuous image \( f(x, y) \) to an **output** 2D image \( g(x, y) \).

In this case, as in the 1D case, we have: \( g(x, y) = T(f)(x, y) \) which is known as the **input-output equation**. For a linear shift-invariant processor, \( g(x, y) = f(x, y) * h(x, y) \), where \( h(x, y) = T(\delta)(x, y) \) is the impulse response. In optical/imaging systems the \( h \) is the PSF of the system.

The image processor should be designed to perform a specific image processing task, e.g., to remove noise from a given image while preserving important image details. Even an analysis task or the measurement of, say, some morphological feature may have an associated “\( T \)”, where the output is the result of a measurement/analysis **process**.
Filtro Pasa-Bajas: Convolución con una **Función de Dispersión Puntual (PSF)** (o un *kernel pasa-bajas*) en Dominio de Frecuencia Espacial (abajo)

Imagen Original

$\downarrow$ FFT

* $\downarrow$ FFT

Imagen Observada

(resolución: $512 \times 512$)

* Jorge Márquez Flores - Image Analysis - Instrumentation - CCADET-UNAM 2012
Filtro Pasa-Altas (bordes y discontinuidades)

Imagen Original $\downarrow$ FFT $\ast$ Kernel Pasa Altas $\downarrow$ FFT $=$ Imagen Observada $\uparrow$ FFT$^{-1}$

(resolución: 512×512)
Filtro Digital de Convolución Espacial
para extraer bordes y discontinuidades

Kernel \( W(m,n) \) de 5×5 con “inhibición lateral”

\[
(f \ast W)(m,n) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l)W(m-k,n-l)
\]

Filtro de bordes por cada canal o sólo en la imagen de intensidad (B/N); más filtro \textbf{mediana} aplicado 3 veces con radio de 2 pixeles:

\[
\text{Nota: se tiene:} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} W(m,n) = 1
\]
Optical and Modulation Transfer Functions (OTF & MTF)

Let us consider linear shift invariant (or spatially invariant) systems (LSI) in 2D. When the inputs and outputs represent a positive quantity, such as the light intensity in imaging systems, then, the Point Spread Function PSF is the impulse response $h(x,y)$ of the system. The frequency response $H(u,v)$ is the Fourier transform of $h(x,y)$: $H(u,v) = \mathcal{F}(h(x,y)) = \mathcal{F}(PSF)$.

For a spatially invariant imaging system its Optical Transfer Function (OTF) is defined as its normalized frequency response:

$$OTF(u,v) = \frac{H(u,v)}{H(0,0)}$$

The OTF of an imaging system is a global measure of its resolution.

The Modulation Transfer Function (MTF) is defined as the magnitude of the OTF:

$$MTF(u,v) = \|OTF(u,v)\| = \frac{\|H(u,v)\|}{\|H(0,0)\|} = \left\| \frac{\mathcal{F}(PSF)}{\mathcal{F}(PSF)}(u,v) \right\|$$

The Phase Transfer Function (PTF) is defined from the argument of the OTF:

$$PTF(u,v) = \exp\left( j \arg(OTF(u,v)) \right), \text{ thus: } OTF(u,v) = MTF(u,v) \cdot PTF(u,v)$$
Modulation Transfer Function (MTF)

MTF of the human visual system. (a) Contrast versus spatial frequency sinusoidal grating; (b) typical MTF plot.

=CSF  Contrast Sensitivity Function (human visual acuity)
Two intensity horizontal profiles of the MTF at heights marked in red.
Cascaded PSF/MTF - Blur Components

- **Process intrinsic blur** (e.g., a secondary radiation source).
- **Diffraction and interference-by-reflexion effects.**
- **Medium blur** (atmospheric, liquid, glass,...) + refractions.
- **Out-of-focus blur**: finite camera aperture and its imperfect shape + imperfect adjustments, finite depth of field, ...
- **Optical aberrations** (geometric distortion, chromatic and spherical aberrations, non-invariant aberrations, etc.).
- **Sensor Blur**.
- **Motion Blur** (Finite exposure time).
- **Anti-aliasing filter** (for spatial-sampling rate below the Nyquist-Shannon rate criterion).
- **Noise**: (external to acquisition system + dark current, shot noise, read noise, quantization, ...).

\[ MTF_{total} = MTF_{process} \cdot MTF_{diffrac} \cdot MTF_{focus} \cdot MTF_{aberrats} \cdot MTF_{sensor} ... \]
Anisotropic Gaussian Kernel 2D
(or bivariate) centered at \((x_0, y_0)\)

\[
f(x, y) = A \exp \left( - \left( \frac{(x-x_0)^2}{2\sigma_x^2} + \frac{(y-y_0)^2}{2\sigma_y^2} \right) \right).
\]

In general, a two-dimensional elliptical Gaussian function is expressed as

\[
f(x, y) = A \exp \left( - \left( a(x-x_0)^2 + 2b(x-x_0)(y-y_0) + c(y-y_0)^2 \right) \right)
\]

where the matrix

\[
\begin{bmatrix}
 a & b \\
 b & c
\end{bmatrix} = \begin{bmatrix}
 \frac{1}{\sigma_x^2} & \frac{\beta}{\sigma_x \sigma_y} \\
 \frac{\beta}{\sigma_x \sigma_y} & \frac{1}{\sigma_y^2}
\end{bmatrix}
\]

is positive-definite.

\[
f(x, y) = A \exp \left( -\frac{1}{2} \left[ (x-x_0)(y-y_0) \right] \begin{bmatrix}
 a & b \\
 b & c
\end{bmatrix} \begin{bmatrix}
 (x-x_0) \\
 (y-y_0)
\end{bmatrix} \right)
\]
\[ f_x(x_1, \ldots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right), \]

\[
|\Sigma| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh.
\]